Exercice 5

I) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ Find the inverse of A if possible

• $\det(A) = 1 \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$

$= -1 + 3(-4) = -13 \neq 0$

$\Rightarrow$ A is invertible

• $\mathrm{cof}(A) = \begin{matrix} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{matrix}$

• $\mathrm{cof}(A) = \begin{bmatrix} -1 & 3 & -6 \\ 3 & -9 & 5 \\ -4 & -1 & 2 \end{bmatrix}$ • $\mathrm{Adj}(A) = \begin{bmatrix} -1 & 3 & -4 \\ 3 & -9 & -1 \\ -6 & 5 & 2 \end{bmatrix}$

• $A^{-1} = \dfrac{1}{\det(A)} \cdot \mathrm{Adj}(A) = \dfrac{1}{-13} \begin{bmatrix} -1 & 3 & -4 \\ 3 & -9 & -1 \\ -6 & 5 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1/13 & -3/13 & 4/13 \\ -3/13 & 9/13 & 1/13 \\ 6/13 & -5/13 & -2/13 \end{bmatrix}$

II) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

. $\det(A) = 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$

$= 1(1) - 2(-3) = \neq 0 \Rightarrow$ A is invertible

. $\mathrm{cof}(A) = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$

. $\mathrm{cof}(A) = \begin{bmatrix} 1 & 3 & -6 \\ -2 & 1 & 5 \\ 2 & -1 & 2 \end{bmatrix}$

. $\mathrm{Adj}(A) = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & -1 \\ -6 & 5 & 2 \end{bmatrix}$

. $A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A)$

$= \frac{1}{7} \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & -1 \\ -6 & 5 & 2 \end{bmatrix}$

. $A^{-1} = \begin{bmatrix} 1/7 & -2/7 & 2/7 \\ 3/7 & 1/7 & -1/7 \\ -6/7 & 5/7 & 2/7 \end{bmatrix}$

III) Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ Find the inverse of A

. $\det(A) = 1\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}$

$= 1(3) - 2(0)$

$= 3 \neq 0 \Rightarrow$ A is invertible

. $\mathrm{Cof}(A) = \begin{bmatrix} \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \end{bmatrix}$

. $\mathrm{Cof}(A) = \begin{bmatrix} 3 & -2 & 2 \\ -0 & 1 & -1 \\ -9 & 5 & -2 \end{bmatrix}$. $\mathrm{Adj}(A) = \begin{bmatrix} 3 & 0 & -9 \\ -2 & 1 & 5 \\ 2 & -1 & -2 \end{bmatrix}$

. $A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & -9 \\ -2 & 1 & 5 \\ 2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ -2/3 & 1/3 & 5/3 \\ 2/3 & -1/3 & -2/3 \end{bmatrix}$

IV) Calculate the determinants of the following matrices

a) $A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

$R_2 + R_1 \to R_2$
$R_3 + R_1 \to R_3$
$R_4 + R_1 \to R_4$

$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$

$\det(A) = -1 \times \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix}$

$= -(1)\left(-2 \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix}\right)$

$= -1(+8 + 8) = -16$

b) $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$

$= -1$

c) $A = \begin{pmatrix} a & a & b & 0 \\ a & a & 0 & b \\ c & 0 & a & a \\ 0 & c & a & a \end{pmatrix}$

$R_2 - R_1 \rightarrow R_2$ $\begin{bmatrix} a & a & b & 0 \\ 0 & 0 & b & b \\ c & 0 & a & a \\ 0 & c & a & a \end{bmatrix}$

$R_4 - R_3 \rightarrow R_4$ $\begin{bmatrix} a & a & b & 0 \\ 0 & 0 & -b & b \\ c & 0 & a & a \\ -c & c & 0 & 0 \end{bmatrix}$

$C_1 + C_2 \rightarrow C_2$ $\begin{bmatrix} a & 2a & b & 0 \\ 0 & 0 & -b & b \\ c & c & a & a \\ -c & 0 & 0 & 0 \end{bmatrix}$

$C_3 - C_4 \rightarrow C_3$ $\begin{bmatrix} a & 2a & b & 0 \\ 0 & 0 & -2b & b \\ c & c & 0 & a \\ -c & 0 & 0 & 0 \end{bmatrix}$

$\det(A) = c \begin{vmatrix} 2a & b & 0 \\ 0 & -2b & b \\ c & 0 & a \end{vmatrix}$

$= c \left(c \begin{vmatrix} b & 0 \\ -2b & b \end{vmatrix} + a \begin{vmatrix} 2a & b \\ 0 & -2b \end{vmatrix} \right) = c\left(cb^2 - 4a^2b\right)$

$=$

d) $A = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ a & 0 & a & 0 & 3 \\ b & a & 0 & a & 0 \\ 0 & b & 0 & 0 & a \end{bmatrix}$

$C_1 - C_3 \to C_1 \quad \begin{bmatrix} -2 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & a & 0 & 3 \\ b & a & 0 & a & 0 \\ 0 & b & 0 & 0 & a \end{bmatrix}$

$C_2 - C_4 \to C_2 \quad \begin{bmatrix} -2 & 0 & 3 & 0 & 0 \\ 0 & -2 & 0 & 3 & 0 \\ 0 & 0 & a & 0 & 3 \\ b & 0 & 0 & a & 0 \\ 0 & b & 0 & 0 & a \end{bmatrix}$

$\det(A) = -2 \begin{vmatrix} -2 & 0 & 3 & 0 \\ 0 & a & 0 & 3 \\ 0 & 0 & a & 0 \\ b & 0 & 0 & a \end{vmatrix} + 3 \begin{vmatrix} 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ b & 0 & a & 0 \\ 0 & b & 0 & a \end{vmatrix}$

$= -2a \begin{vmatrix} -2 & 0 & 0 \\ 0 & a & 3 \\ b & 0 & a \end{vmatrix} + 3 \times 3 \begin{vmatrix} 0 & -2 & 3 \\ b & 0 & a \\ 0 & b & 0 \end{vmatrix}$ -

$= (-2a \times (-2)) \begin{vmatrix} a & 3 \\ 0 & a \end{vmatrix} + 9(-b) \begin{vmatrix} 0 & 3 \\ b & a \end{vmatrix}$

$= 4a(a^2) - 9b(-3b) = 4a^3 + 27b^2$

e) $A = \begin{pmatrix} a+b & a & a & \cdots & a \\ a & a+b & a & \cdots & a \\ \vdots & & \ddots & & \vdots \\ a & \cdots & & \cdots & a+b \end{pmatrix}$

$c_i - c_1 \to c_i$
$i = 2 \ldots n$

$$\begin{pmatrix} a+b & -b & -b & \cdots & -b \\ a & b & 0 & \cdots & 0 \\ \vdots & 0 & b & & \vdots \\ \vdots & \vdots & & \ddots & \\ a & 0 & \cdots & \cdots & b \end{pmatrix}$$

$R_1 + R_2 + \cdots R_n \to R_1$

$$\begin{pmatrix} na+b & 0 & 0 & \cdots & 0 \\ a & b & & & \vdots \\ \vdots & & \ddots & & \\ \vdots & & & & 0 \\ a & \cdots & & \cdots & b \end{pmatrix}$$

$\det(A) = (na+b)\, b^{n-1}$

V) $A = \begin{bmatrix} e^t & \cos t & +\sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}$ Find the inverse of A

$R_1 + R_3 \to R_3$ $\begin{bmatrix} e^t & \cos t & +\sin t \\ e^t & -\sin t & \cos t \\ 2e^t & 0 & 0 \end{bmatrix}$

$\det(A) = e^t \begin{bmatrix} \cos t & +\sin t \\ -\sin t & \cos t \end{bmatrix}$

$$\det(A) = 2e^t\,(\overbrace{\cos^2 t + \sin^2 t}^{1})$$
$$= 2e^t > 0 \implies A \text{ is invertible}$$

$$\mathrm{cof}(A) = \begin{bmatrix} \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} & -\begin{vmatrix} e^t & \cos t \\ e^t & -\sin t \end{vmatrix} & \begin{vmatrix} e^t & -\sin t \\ e^t & -\cos t \end{vmatrix} \\ -\begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix} & \begin{vmatrix} e^t & \sin t \\ e^t & -\sin t \end{vmatrix} & -\begin{vmatrix} e^t & \cos t \\ e^t & -\cos t \end{vmatrix} \\ \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} & -\begin{vmatrix} e^t & \sin t \\ e^t & \cos t \end{vmatrix} & \begin{vmatrix} e^t & \cos t \\ e^t & -\sin t \end{vmatrix} \end{bmatrix}$$

$$[\mathrm{cof}(A)]^T = \mathrm{Adj}(A) = \begin{bmatrix} 1 & e^t(\sin t + \cos t) & e^t(\sin t - \cos t) \\ 0 & -2e^t \sin t & 2e^t \cos t \\ 1 & e^t(\sin t - \cos t) & -e^t(\sin t + \cos t) \end{bmatrix}^T$$

$$A^{-1} = \frac{1}{2e^t}\begin{bmatrix} 1 & 0 & 1 \\ e^t(\sin t + \cos t) & -2e^t \sin t & e^t(\sin t - \cos t) \\ e^t(\sin t - \cos t) & 2e^t \cos t & -e^t(\sin t + \cos t) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2e^t} & 0 & \frac{1}{2e^t} \\ \frac{1}{2}(\sin t + \cos t) & -\sin t & \frac{1}{2}(\sin t - \cos t) \\ \frac{1}{2}(\sin t - \cos t) & \cos t & -\frac{1}{2}(\sin t + \cos t) \end{bmatrix}$$

VI) $A = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{bmatrix}$ Find inverse of A

$$\det(A) = 1 \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} + t \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}$$

$$= 1(2) + t(t^2)$$

$$= 2 + t^3 .$$

$$\mathrm{cof}(A) = \begin{matrix} \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 2t \\ t & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ t & 0 \end{vmatrix} \\ -\begin{vmatrix} t & t^2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & t^2 \\ t & 2 \end{vmatrix} & -\begin{vmatrix} 1 & t \\ t & 0 \end{vmatrix} \\ \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} & -\begin{vmatrix} 1 & t^2 \\ 0 & 2t \end{vmatrix} & \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} \end{matrix}$$

$$\mathrm{cof}(A) = \begin{matrix} 2 & 2t^2 & -t \\ -2t & 2-t^3 & t^2 \\ t^2 & -2t & 1 \end{matrix}$$

$$\mathrm{Adj}(A) = \begin{matrix} 2 & -2t & t^2 \\ 2t^2 & 2-t^3 & -2t \\ -t & t^2 & 1 \end{matrix}$$

$$A^{-1} = \frac{1}{2+t^3} \begin{bmatrix} 2 & -2t & t^2 \\ 2t^2 & 2-t^3 & -2t \\ -t & t^2 & 1 \end{bmatrix}$$